PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Non-commutative geometry and quantum groups

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A search for the unification of quantum theory and gravity has forced mathematical physicists to re-evaluate the meaning of geometry itself. The surprising answer has led to an explosion of research papers, a vast collection of examples, and to revolutions in at least three branches of pure mathematics. It offers insights into the origin of the universe and the nature of physical reality.

Keywords: cosmology; curvature; Fourier; Heisenberg; Hopf; knots

1. Introduction

Theoretical physics is something of a quest, a search for a philosophical and mathematical formulation of the laws that govern nature that is both consistent and complete. Both Einstein's formulation of gravity as the geometry of space-time, and the discovery of quantum theory, were enormous conceptual leaps made in the early decades of the 20th century. However, these two theories are not really very compatible (one may recall Einstein's famous dislike of quantum mechanics) with the result that a complete unified theory has remained something of a 'holy grail'. Recent developments may be starting to change that.

First of all, it should not be forgotten that classical geometry, which Einstein used to formulate gravity, grew out of our macroscopic intuition and experiences, such as the trajectories of particles. On a subatomic scale the world simply is not like that. There are no precise trajectories, everything is 'fuzzy' or 'wave-like'. There is therefore no logical reason to think that one can cling to classical geometry and still develop a theory that truly unifies both quantum effects and gravity. It is more likely that both will need to be modified in the process of developing a unified theory. Such a theory could be viewed as a generalization of geometry compatible with quantum theory (or vice versa) and could be expected to be as radically different as the leap from Euclidean to non-Euclidean geometry. If history is any guide, it is further likely that the conceptual and mathematical structure of the required 'quantum' geometry would itself be a guide to finding such a complete theory of nature. One might reasonably regard the understanding of the mathematics of quantum geometry as a prerequisite.

This has not stopped physicists speculating about quantum gravity, black holes and the birth of the universe (regimes where both quantum and gravitational effects are strong), but those speculations can never be anything more than some kind of approximation, breaking down at some point. It has also not stopped physicists developing more and more complicated theories, some with quite elegant formulations

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(such as string theory), but nevertheless based on conventional ideas of geometry and quantum theory. The truth of the matter is that physicists have not yet had at their disposal any actual mathematical framework of a more radical 'quantum' geometry which might be usable for any truly unified theory.

This conceptual problem has begun to be addressed in the last two decades of the 20th century, and could well bring us within reach of the 'holy grail' in the next one. What *is* the correct geometry of the quantum world? First of all, note that in the classical mechanics that describes the macroscopic world, we work with coordinates like x, p (the position and momentum of particles). In quantum mechanics these become operators $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{xp}$ no longer equals $\mathbf{px}$ (the operators do not commute). This has the interpretation that it matters which you measure first, $\mathbf{x}$ or $\mathbf{p}$, and this in turn is related to the famous Heisenberg uncertainty principle, that you cannot measure both of them accurately at the same time. This non-commutativity of position and momentum 'coordinates' is, therefore, completely fundamental to quantum theory as we know it. Hence, to have a fully geometrical picture in the quantum world, one needs in fact a new foundation of geometry based on non-commuting operators in place of usual coordinates. This, broadly, is 'quantum' or 'non-commutative' geometry. Note that in conventional quantum mechanics, the components of the position $\mathbf{x}$ still commute among themselves, but as soon as one has a more general geometry to include the $\mathbf{p}$, there is no need to assume this either, which is one of the novel theoretical predictions coming out of non-commutative geometry. Such a prediction, if observed, would amount to a new physical effect or 'force' in nature. To put it in perspective, non-commutativity of the components of the appropriate momentum $\mathbf{p}$ is already observed in nature (although not usually discussed in this manner): it corresponds to curvature in position space or, very roughly speaking, to gravity. The new physical effect would be reciprocal to this under an interchange of position and momentum. And all three effects would be bound up in the overall structure of the combined $\mathbf{x}$ and $\mathbf{p}$.

Today, what is seen is mainly a rich collection of examples of non-commutative geometry, the most accessible of which are called 'quantum groups'. Even without the above philosophical considerations, quantum groups' structures have thrust themselves forward in the last two decades in a variety of settings in mathematics and mathematical physics. They have already revolutionized at least three branches of pure mathematics: notably, knot theory, Poisson geometry and representation theory. More than any grandiose mathematical proposals for non-commutative geometry, the richness of the examples and applications, and the foundation of the theory on them, is the most convincing evidence that mathematical physics is on the right track. Figure 1 shows the explosive growth of this theory,† not including many additional papers on related topics.

Also emerging are some powerful tools, notably a new kind of 'braided algebra' as a mathematical calculus for non-commutative geometry. Instead of manipulating symbols as in usual algebra, algebraic operations are 'wired up', much as the processing modules of a computer are literally wired up. Information flows along the 'wires' except that (in braided algebra) the under- and over-crossings of wires are non-trivial (and generally distinct) operations. This too is inspired by quantum physics, where matter particles or 'fermions' (such as electrons) have the unusual

† Data complied from BIDS: published papers since 1981 with title or abstract containing 'quantum group*', 'Hopf alg*', 'non-commutative geom*', 'braided categ*', 'braided group*', 'braided Hopf*'.

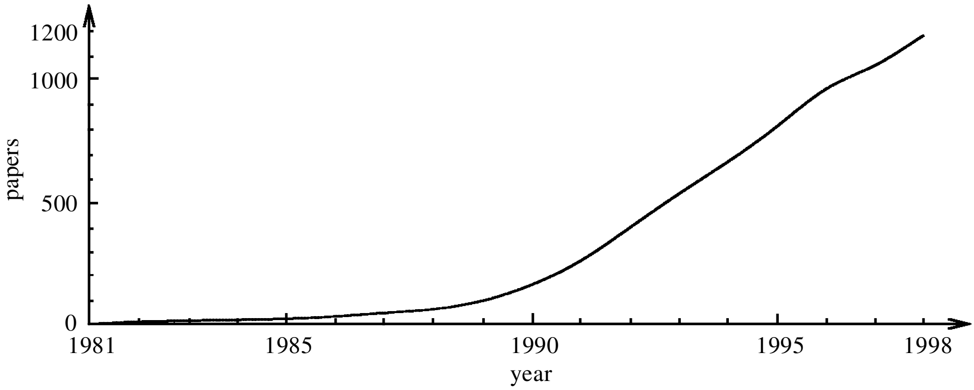

Figure 1. Growth of research papers on quantum groups.

feature that when they are exchanged in a physical process there is an additional minus sign that goes along with the transposition. It is this minus sign that is the origin of the famous Pauli exclusion principle: that two electrons cannot be in exactly the same state. In braided algebra this minus sign associated to 'fermionic statistics' is replaced by a more general matrix operation associated instead to more general 'braid statistics'. The reader may notice a vague similarity between the Pauli exclusion principle here and the Heisenberg uncertainty principle above. In fact, the idea of non-commutativity as in quantization and the idea of braid statistics are intimately related in non-commutative geometry, as we will see.

To get an immediate flavour of what these ideas mean in practice, consider the following elementary computation. For a polynomial function f in one variable, define differentiation by

$$f'(y) = (x^{-1}(f(x+y) - f(y)))_{x=0}. \tag{1.1}$$

If $xy = yx$ is assumed in making the calculation, one obtains the usual Newtonian differentiation. But if we suppose $yx = qxy$ in computing the right-hand side, for some parameter q, we obtain

$$f'(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{1.2}$$

This is the celebrated 'q-deformed derivative', so called because it tends to the usual derivative as $q \to 1$. Although known to mathematicians in a different context in 1908 (Jackson 1908), such q-derivatives have their natural place in the geometry of quantum groups. We also see by this example that non-commutativity leads to a kind of 'finite difference' or discretization, which is, therefore, a general feature of the differential geometry of the quantum world.

Finally, returning to theoretical physics, there emerges an interesting philosophical principle visible only in quantum geometry: a self-duality between quantum theory and gravity. The duality here is a kind of 'Fourier transform', and a remnant of it in conventional quantum mechanics is 'wave–particle duality'. We will conjecture that it provides a new foundation for physics in the 21st century.

2. Algebras with everything

If we are serious about the unification of quantum theory and geometry, it should be self-evident that we must first cast both of them in the same language. The language required is that of algebra. For our purposes, an algebra means an object A equipped with a product sending $a, b \in A$ to $a \cdot b \in A$ in a bilinear manner. One writes the product $\cdot$ as a map

$$A \otimes A \to A, \tag{2.1}$$

where the tensor product symbol $\otimes$ signifies that the product is linear with respect to each of its two inputs (we are assuming that A is a vector space so that we can add elements of A and scale them). In addition, we demand associativity, which is that the two compositions

$$\begin{array}{ccccc} & & A \otimes A & & \\ & {\scriptstyle \cdot\otimes \mathrm{id}} \nearrow & & \searrow {\scriptstyle \cdot} & \\ A \otimes A \otimes A & & & & A \\ & {\scriptstyle \mathrm{id}\otimes\cdot} \searrow & & \nearrow {\scriptstyle \cdot} & \\ & & A \otimes A & & \end{array} \tag{2.2}$$

coincide, where id denotes the identity map. This is a fancy way of saying that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c \in A$. We also require a unit $1 \in A$ such that $a \cdot 1 = a = 1 \cdot a$ for any a, which too can be expressed in terms of maps. This abstract approach will be relevant later on. In practical terms, however, we are typically interested in *particular* elements in the algebra that generate the others. In this case, the algebra amounts to specifying equations that these elements obey among themselves and various rules for their manipulation. This is closer to what one means colloquially by 'algebra', in contrast to the definition above.

We turn now to our task. First of all, quantum mechanics, as usually formulated, is expressed in terms of wave functions (x) obeying Schrödinger's equation. It might be argued, therefore, that even quantum mechanics assumes a usual space whereon the wave functions live. However, the points in this space are no more than the possible macroscopic values that can result from a position measurement (with probability $| \ (x)|^2$). They reflect a particular range of results for questions one can ask about the quantum system but are not the quantum system itself. There are still operators, such as $\mathbf{x}$, $\mathbf{p}$, and, in the usual interpretation of quantum mechanics, any self-adjoint operator on the set (in fact, Hilbert space) of wave functions is a valid 'observable' or question that one might ask of the system. The only algebra in this conventional picture is that of all (bounded) operators on the set of wave functions, which is the same algebra for all quantum systems. (Instead, the content of the quantum system rests in the choice of Hamiltonian operator.) In quantum statistical mechanics, one goes slightly further and considers as a 'state' a density matrix or convex linear combination of projection matrices associated to wave functions $\ _i(x)$ with weighting s_i, where $\sum s_i = 1$, but the structure is otherwise the same. These comments also apply to quantum field theory where the role of x is played by a field, but again it is a field on conventional space-time and again it merely reflects a choice concerning what kinds of questions one is asking about the quantum theory. In effect, we project our preconceptions and intuition derived from macroscopic geometry onto quantum theory by focusing on particular states and observables.

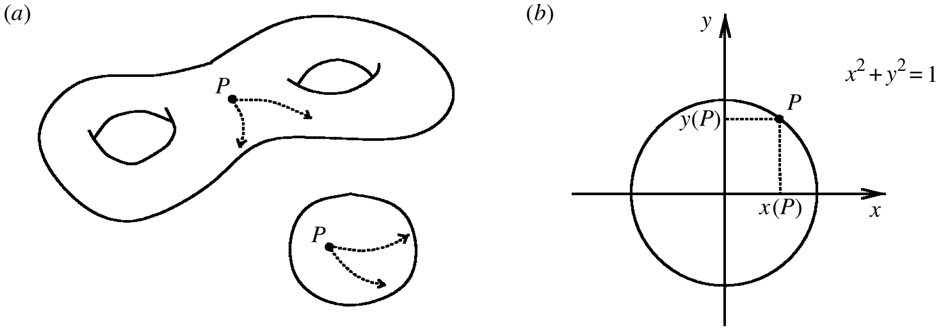

Figure 2. (*a*) Two-dimensional geometry visualized in three dimensions; (*b*) coordinate functions.

There is, however, a more sophisticated point of view of quantum theory which brings out more of its intrinsic quantum structure. This is the C^*-algebraic point of view, which was applied particularly in mathematical physics by G. W. Mackey and I. Segal in the 1960s (see Bratteli & Robinson 1979). Here, the 'observables' of the quantum system are the elements of *any* abstract algebra A with certain properties (such as an involution and a norm). A 'state' in this picture is any positive linear functional $\phi : A \to \mathbb{C}$. It assigns to an observable $a \in A$ its expectation value $\phi(a)$. This includes the standard version of quantum (statistical) mechanics as a special case, with A the algebra of all operators on a Hilbert space of states and

$$\phi(a) = \sum_i s_i \langle \;{}_i|a|\;{}_i\rangle \tag{2.3}$$

in the usual notations in quantum theory. Our more general approach, however, allows for the possibility that not all operators are allowed observables: *A might be only a subalgebra of all operators.* With this caveat, the new approach is essentially equivalent to the usual one. A theorem of Gelfand and Naimark asserts that for any A and choice of vacuum state ϕ_0, one can build up a Hilbert space and identify A as a subalgebra of its operators. In this way, quantum mechanics becomes the study of an algebra A and positive linear functionals ϕ on it.

The particular algebra A of observables allowed in the quantum system is called the *kinematic structure* and it is exactly this that is missed in the conventional point of view. Just as every manifold or curved space can be viewed concretely as embedded in flat space of a suitably high dimension (see figure 2*a*), in the same way, the subalgebra that A 'carves out' in the algebra of all operators is the 'quantum geometry' or kinematic structure of the model. Put another way, the algebras in quantum theory are many and more varied than just the algebra generated by the Heisenberg commutation relations

$$\mathbf{xp} - \mathbf{px} = \mathrm{i}\hbar, \tag{2.4}$$

which (in one guise or another) is the only algebra usually encountered in a first course on quantum mechanics. Here, $\hbar$ is Planck's constant.

Next, we cast usual concepts of geometry too in an algebraic form. This is 19th-century work but was extensively developed early on in the 20th century by Zariski, Grothendieck and others. Thus, in algebraic geometry we work with the equations

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satisfied by the coordinates on spaces, rather than thinking of points themselves. For example, a circle is the equation

$$x^2 + y^2 = 1, \tag{2.5}$$

by which we mean it is the set of points P in the plane for which the coordinate functions $x(P)$ and $y(P)$ obey this equation. Here x, y are functions that assign to P in the plane its two coordinates (as shown in figure 2*b*). The equation (2.5), however, makes sense without reference to points, but instead with reference to the algebra generated by x, y under pointwise multiplication and addition

$$(xy)(P) = x(P) \cdot y(P), \qquad (x + y)(P) = x(P) + y(P), \tag{2.6}$$

for all points P. Given a set X of points, we can always consider its algebra of functions $\mathbb{C}[X]$ (often generated by particular coordinate functions such as x, y in the above example). But note that such algebras are necessarily commutative,

$$(xy)(P) = x(P)y(P) = y(P)x(P) = (yx)(P), \tag{2.7}$$

for all points P, i.e. $xy = yx$ as an intrinsic feature of such algebras.

Conversely, a theorem also due to Gelfand and Naimark says that every commutative C^*-algebra A arises as the algebra of functions on some (topological) space. Indeed, the points P can be identified with the 'states' or positive linear functionals $A \to \mathbb{C}$, sending any function x to $x(P)$. Thus, there is, in fact, quite a bit of common ground between quantum theory and classical geometry. Both are described by algebras and positive linear functionals. The main difference is that classical geometry corresponds firmly and intrinsically to A commutative.

However, although this point of contact was striking, until recently it has had relatively little impact because the two lines of development went on in quite different directions. C^*-algebras and other aspects of operator theory were extensively developed in connection with quantum theory, but the 'non-commutative geometrical view' was largely limited to extending topological ideas, such as measures and K-theory, to the case of non-commutative operator algebras (Connes 1994). There have been occasional links with physics (e.g. the quantum Hall effect) and promising attempts at deeper aspects of non-commutative geometry (notably, Connes' 'spectral geometry'), but the full *infrastructure* of non-commutative differential geometry—coordinate charts, metric, connections, curved spaces, etc.—cannot easily be approached in this way. Meanwhile, algebraic geometry too was pushed forward by several great mathematicians (often in connection with number theory and the theory of algebraic groups and their representations). By now it includes almost all familiar geometrical notions, including those mentioned above, via algebraic tools such as 'sheaf theory' and 'etale cohomology'. However, most of that development strongly assumes commutativity and does not work at all well when the algebras are non-commutative.

What happened to change all this and bring about a convergence of these two traditions? In practical terms it was the discovery of 'quantum groups' in the mid-1980s. These were a rich vein of examples of non-commutative algebras with natural 'generators' (coordinates) and a clear geometrical structure. Hitherto, there had been very few concrete algebras to play with, namely the Heisenberg algebra (2.4) and variants of it, such as the 'non-commutative torus'; and one cannot found a theory of geometry on one (albeit beautiful) example. Moreover, quantum groups have many applications in their own right, bringing many potential benefits of non-commutative geometry beyond our thoughts about quantum gravity. The application of quantum groups and the geometrical point of view in quantum chemistry, for example, remain largely unexplored.

3. Quantum groups

Quantum groups are the simplest convincing examples of non-commutative geometry. There are two main classes of these at the moment: one as 'generalized transformations' arising in solid state physics (Drinfeld 1987; Jimbo 1985); and another arising from toy models of quantum gravity (Majid 1988). We shall briefly describe both.

By way of preamble, let us recall that a group is a set with a product law, a unit and an inversion operation. Of course, groups have many applications, notably in crystallography. The groups with a clear geometrical structure, however, are smooth continuous ones, the so-called Lie groups. They typically arise as transformation groups, for example, the group SO_3 is the group of rotations in three dimensions. If you make two rotations in succession, the composition is a third rotation about some other axis. Similarly, the group SL_2 is that of 2×2 matrix transformations of determinant 1. Moreover, infinitesimal such transformations form 'Lie algebras', and all simple Lie algebras over $\mathbb{C}$ (the basic building blocks in Lie theory) were classified at the start of the 20th century; their beautiful properties are still being studied today. Such objects have featured since the 1950s in elementary particle physics as the symmetries involved in the fundamental forces of nature. We want to give some equally beautiful generalizations of these objects. By the way, it is possible to skip this entire section if you prefer not to go into the mathematics.

Consider first of all the 'quantum plane' A_q^2. This is the algebra generated by variables x, y but with the relations

$$yx = qxy, \tag{3.1}$$

instead of commutativity. Here, q is a non-zero numerical parameter. When $q = 1$, we can consider x, y as the coordinates on an actual plane as we did above, but when $q \neq 1$, the algebra is non-commutative and, hence, there is no usual space underlying it. We also have higher-dimensional quantum spaces of many kinds depending on the relations and parameters. In particular, the quantum group $SL_{q,2}$ has generators a, b, c, d with the six relations

$$\left.\begin{array}{lll} ba = qab, & dc = qcd, & ca = qac, \\ db = qbd, & bc = cb, & ad - da = (q^{-1} - q)bc, \end{array}\right\} \tag{3.2}$$

which describe a four-dimensional q-space (they become the relations of commutativity when $q = 1$), and the additional relation

$$ad - q^{-1}bc = 1, \tag{3.3}$$

which sets the 'q-determinant' to 1.

There is nothing much that need concern us about the exact form of the above relations. Apart from giving the expected relations among coordinates when $q = 1$, their exact form is largely dictated by requiring that various properties of 2×2

matrices and their action on vectors go through even when $q \neq 1$. Thus, if x, y generate a quantum plane, then

$$x' = ax + by, \qquad y' = cx + dy \tag{3.4}$$

obey the relations $y'x' = qx'y'$ of the quantum plane as well. In mathematical terms, this 'quantum transformation' is an algebra map $A_q^2 \to SL_{q,2} \otimes A_q^2$. Note that the arrow goes in the reverse direction to what one might have expected if one thought that the matrix was being combined with a vector to give another vector. This is because the a, b, c, d, etc., are not actual numbers but abstract symbols having values only in a representation of the algebras concerned.

To complete the picture here, we need to understand how the group structure itself is expressed in our algebraic language. In the above example, the ability to multiply two matrices to get a third matrix corresponds to the assertion that if a', b', c', d' are a second mutually commuting copy of $SL_{q,2}$, then

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{3.5}$$

obeys the same relations. In mathematical terms, the group law is expressed as an algebra map $\Delta : SL_{q,2} \to SL_{q,2} \otimes SL_{q,2}$ called the 'co-product', where

$$\Delta a = a \otimes a + b \otimes c, \qquad \Delta b = b \otimes d + a \otimes b,$$
$$\Delta c = c \otimes a + d \otimes c, \qquad \Delta d = d \otimes d + c \otimes b,$$

corresponds entry by entry to the multiplication of matrices (3.5). One can write it more compactly as

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.6}$$

Likewise, the unit matrix can be viewed as a map $\epsilon : SL_{q,2} \to \mathbb{C}$ called the 'co-unit', and the matrix inversion can be viewed as a map $S : SL_{q,2} \to SL_{q,2}$ called the 'antipode' or 'co-inverse'. These constructions can also be cast into the setting of operators and C^*-algebras as in § 2 (see Woronowicz 1987).

The formal definition of a 'quantum group' or 'Hopf algebra' is:

(a) an algebra H with a unit element 1;

(b) a co-algebra structure on H with a co-product Δ and a co-unit ϵ;

(c) the above compatible with each other and with an antipode S.

A co-algebra is just like an algebra (2.1)–(2.2) but mapping in the opposite direction. Thus there is a map Δ sending $H \to H \otimes H$ that is *co-associative* in the sense of (2.2) *with the arrows reversed* (and Δ in place of the product). And, having both structures, the axioms of a quantum group are invariant under the reversing of arrows, interchanging products and co-products, etc. Following from this, for every quantum group H (with some technical caveats), there is another quantum group H^*. The required H^* here is essentially the space of linear maps $H \to \mathbb{C}$ endowed with a certain Hopf algebra structure determined by that of H (and vice versa). Note that if H were a quantum system, then, as explained in § 2, H^* is where the states would live. We will return to this later on.

For example, dual to the above matrix quantum group $SL_{q,2}$ is the enveloping algebra quantum group $U_q(sl_2)$. It can be described explicitly as generated by h, x_+, x_-, say, with the relations and co-product

$$x_+x_- - x_-x_+ = \frac{q^h - q^{-h}}{q - q^{-1}}, \qquad hx_\pm - x_\pm h = \pm 2x_\pm,$$

$$\Delta h = h \otimes 1 + 1 \otimes h, \qquad \Delta x_\pm = x_\pm \otimes q^{h/2} + q^{-h/2} \otimes x_\pm.$$

The mathematically minded reader may recognize here that, as $q \to 1$, the relations

$$[x_+, x_-] = h, \qquad [h, x_\pm] = \pm 2x_\pm \tag{3.7}$$

of the Lie algebra sl_2. We will discuss Lie algebras more precisely in § 5, but, for our present purposes, we understand $[x_+, x_-]$ as $x_+x_- - x_-x_+$, etc., in which case, the algebra generated by (3.7) is called the 'enveloping algebra' $U(sl_2)$ of the Lie algebra sl_2. Its co-product $\Delta x_\pm = x_\pm \otimes 1 + 1 \otimes x_\pm$, etc., corresponds, in physics, to the addition law for angular momentum; the action on a tensor product system is the sum of the actions on each part. Similarly, one has quantum groups $U_q(g)$ associated to every simple Lie algebra g as a deformation of its enveloping algebra $U(g)$ (Drinfeld 1987; Jimbo 1985). It is worth noting that although the concept of a quantum group was introduced by H. Hopf in the 1940s, no significant classes of examples going truly beyond usual groups or Lie algebras were known until the mid-1980s, when the modern theory arrived.

The second class of quantum groups arose from the point of view of quantum mechanics combined with gravity (Majid 1988). The simplest of these is the so-called 'Planck scale' quantum group $\mathbb{C}[\mathbf{x}] \blacktriangleright\!\!\triangleleft \mathbb{C}[\mathbf{p}]$, generated by two variables $\mathbf{x}$, $\mathbf{p}$ with

$$\mathbf{xp} - \mathbf{px} = \mathrm{i}\hbar(1 - \mathrm{e}^{-(c^2/MG)\mathbf{x}}), \qquad \Delta\mathbf{x} = \mathbf{x} \otimes 1 + 1 \otimes \mathbf{x}, \qquad \Delta\mathbf{p} = \mathbf{p} \otimes \mathrm{e}^{-(c^2/MG)\mathbf{x}} + 1 \otimes \mathbf{p}.$$

where G is Newton's gravitational constant, c is the speed of light, and M is a remaining parameter. The co-product here corresponds as in (3.5) to the multiplication of matrices of the form

$$\begin{pmatrix} \mathrm{e}^{-(c^2/MG)\mathbf{x}} & 0 \\ \mathbf{p} & 1 \end{pmatrix}.$$

We therefore obtain the coordinate algebra $\mathbb{C}[X]$ of the classical group X of such matrices when $\hbar \to 0$, which is the classical mechanical system of which the Planck-scale quantum group is the quantization. The non-triviality of the group X corresponds to a degree of curvature in the classical system. Meanwhile, in the limit $G \to 0$, we obtain the usual Heisenberg algebra (2.4) of quantum mechanics in situations where we can consider that $\mathbf{x} \gg 0$. In the general case, we have both quantum and gravitational features! In particular, a particle with Hamiltonian $\mathbf{p}^2/m$ starting out at $\mathbf{x} \gg 0$ and moving towards the origin will, at large distances, behave like a usual free particle. As the particle approaches the origin, however, it will go more and more slowly and will, in fact, take an infinite amount of time to reach the origin. This is quite similar to the behaviour of a particle as it moves towards the event

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horizon of a black hole of mass M. This second class of quantum groups is self-dual, in that the dual quantum group has a similar form.

It should be stressed that this is only a toy model; some further ideas will surely be needed to really describe a black hole combined with quantum mechanics in this way. On the other hand, this second class of quantum groups also includes one associated to every complex simple Lie algebra g, the quantum group $\mathbb{C}[G^*] \blacktriangleright\!\!\triangleleft U(g)$. Here, g^* is a certain other Lie algebra dual to g (see below) and G^* is its associated group. It is interesting that this family of quantum groups is just as general and natural as the $U_q(g)$ family, although less well studied to date. A puzzle here is that, while both of these quantum groups spring from the same data—a simple Lie algebra g and certain structure on it—there is, to date, no known direct relation between them:

$$\begin{array}{ccc} & g & \\ \swarrow & & \searrow \\ U_q(g) & ? \rightarrow & \mathbb{C}[G^*] \blacktriangleright\!\!\triangleleft U(g). \end{array}$$

4. Some beautiful applications

Before proceeding, we turn to some of the reasons to be interested in quantum groups in their own right. In fact, there are diverse areas of mathematics that were both revolutionized and related to each other by quantum groups.

(a) *Knot theory*

This is an age-old problem, considered, for example (without much success), by the great 19th-century physicist Kelvin: how can one tell if a knot, such as the trefoil knot in figure 3*a*, is trivial, i.e. can be unknotted into a circle (other than by trial and error). Similarly, how can one tell if two knots are really the same? One would like something computable on any knot that has the same value if (and, ideally, only if) the knots are the same. The theory of such 'knot invariants' made slow progress for most of the century but was revolutionized in the mid-1980s, about the time that quantum groups were being discovered and in correlation with that. Specifically, Jones (1985) showed how to construct a polynomial in a variable q, q^{-1} for any knot, often able to distinguish knots. He was motivated by the theory of solvable lattice models (see a later section) and, through this, his polynomial knot invariant turned out to be connected with a representation of the quantum group $U_q(sl_2)$. Developing that, Reshetikhin & Turaev (1990) obtained polynomial knot invariants associated to the representations of every quantum group $U_q(g)$. A problem that had seemed intractable for most of a century suddenly had a vast number of solutions.

We will briefly describe these knot invariants. In fact, Shakespeare was perhaps on the right track when Viola in *Twelfth Night* says

> O time, thou must untangle this, not I
> It is too hard a knot for me t'untie.

Thus, rather than thinking of the knot in three-dimensional space as in figure 3*a* (a problem apparently too hard to resolve), we interpret the vertical axis as time and read the knot as describing the trajectories of particles V and antiparticles V^* flowing down the page (figure 3*b*). Here, the downward arcs are the particles flowing

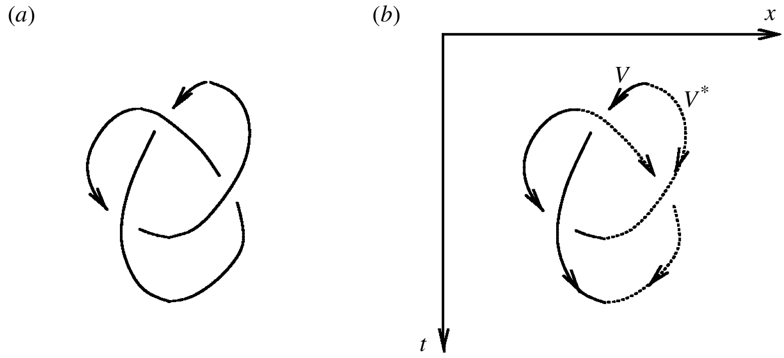

Figure 3. Trefoil knot (a) and construction (b) of its invariant.

down while the upward arcs in figure 3a are regarded in figure 3b as antiparticles flowing down rather than particles flowing up. (It was the physicist R. Feynman who famously remarked that an antiparticle is just a particle going backwards in time.) When one particle passes over or under another, we apply some kind of operation R according to the flavour of the crossing. In this way, as time proceeds it 'scans' the knot from top to bottom, creating particles as needed, interacting them at the crossings, and finally fusing particles and antiparticles as needed. The total process is computable and, very roughly speaking, is the knot invariant as a function of any parameter q on which R might depend.

In effect, rather than viewing the knot in three dimensions, we regard it instead as a process for the interaction of particles and antiparticles moving in one space and one time dimension, according to how the original knot looked on the page. We need to know that if we drew the knot from a different angle and did our process from that point of view, we should get the same answer. We also need to know that if we distort the knot without cutting it then we get the same answer, all of which depends on choosing R carefully. The latter part of the problem can be reduced mainly to the braid relation in figure 4. These braids are topologically the same, so replacing one with the other in a complicated knot would not change it. Therefore, we require that the corresponding operations R should give the same total process on three particles. This is the so-called *quantum Yang–Baxter equation* for R also shown in figure 4. Here we suppose that the states of the particle V are described by a vector space and write the interaction process R sending $V \otimes V \to V \otimes V$ concretely as a matrix with four indices (and we use Einstein's summation convention that repeated indices are to be summed over). We also require similar relations where some strands are antiparticles. The remainder of the problem can be focused mainly on the observation that a harmless twist in the knot can appear untwisted when viewed from a different angle, so that the number of crossings themselves can change. In typical examples, the matrices R, while not invariant under such harmless twists, usually change in a simple way that can be compensated for by hand. Actually, what one obtains in this way is not exactly a knot invariant but an invariant of ribbons or framed knots. Apart from these subtleties, these are the main constraints that R have to satisfy.

As for the choice of R, a particle in physics is not merely a vector space but typically forms a representation V of a Lie algebra g, and its conjugate the dual representation V^*. When particles are interchanged, one usually has either an exchange

$$R^{i}{}_{a}{}^{k}{}_{b}R^{a}{}_{j}{}^{m}{}_{c}R^{b}{}_{l}{}^{c}{}_{n} = R^{k}{}_{b}{}^{m}{}_{c}R^{i}{}_{a}{}^{c}{}_{n}R^{a}{}_{j}{}^{b}{}_{l}$$

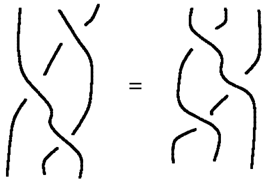

Figure 4. Yang–Baxter equation for matrix R.

$R^{i}{}_{j}{}^{k}{}_{l}(\lambda)$

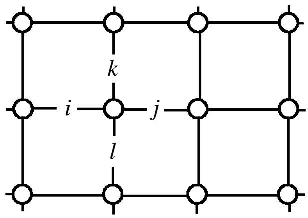

Figure 5. Solvable lattice model in statistical mechanics has weight R at each vertex.

factor $R=-1$ (for fermionic particles like the electron) or the trivial exchange $R=1$ (for bosonic particles like the photon). Neither of these choices gives interesting knot invariants, but when we look instead at representations of the quantum groups $U_q(g)$, we find a much more non-trivial matrix R (depending on q) whenever two representations of the quantum group are exchanged. We can then proceed as with the heuristic particle picture above but with V a representation of the quantum group $U_q(g)$ and V^* its dual representation. They are created together as the canonical element of $V\otimes V^*$ (or a certain other element of $V^*\otimes V$) and are fused by the evaluation map $V^*\otimes V\to\mathbb{C}$ (or a certain other map $V\otimes V^*\to\mathbb{C}$). The result is a function of q, and this, more precisely, is the construction of the knot invariants from quantum groups.

We note in passing that there are also potential biological applications, for example for detecting the allowed states of tangled or knotted DNA molecules.

(*b*) *Solvable lattice models and integrable systems*

It is a testament to the fertile role of physics in pure mathematics that some of the main ingredients above actually came out of solid-state physics in two dimensions. Thus, in statistical mechanics one has a large collection of distinct states of the system and studies its bulk properties through the partition function, a certain weighted sum over the states. For example, consider the model of a crystal in figure 5, where a state is an assignment of bonds throughout the lattice. We write the *Boltzmann weight* at each vertex as the entry of a matrix R according to the value of the bonds around the vertex. The partition function is

$$Z(\lambda)=\sum_{\text{states}}\prod_{\text{vertices}}R^{i}{}_{j}{}^{k}{}_{l}(\lambda), \tag{4.1}$$

where $ijkl$ are the values of the bonds in the given state surrounding the given vertex. We suppose the weight depends on a parameter λ.

Working in a different (but broadly equivalent) setting, Baxter (1982) described conditions on R that allowed for the partition function to be computed explicitly using a 'corner transfer matrix method'. The resulting functions often had beautiful connections with number theory and the theory of modular forms (not connected with A. Wiles's recent proof of Fermat's theorem, but in the same general ballpark). The required conditions were that R depends on a parameter λ and obeys a parametrized version of the Yang–Baxter equation in figure 4. So the key idea behind the knot invariants also makes these models solvable. Later on, I. Sklyanin, L. D. Faddeev and others recast the corner transfer matrix method more algebraically through a quantum group with a matrix of generators and relations defined by $R(\lambda)$. The parameter-free versions of these led to quantum groups such as $SL_{q,2}$ and $U_q(sl_2)$. The relevant model for the latter (the so-called XXZ model) consists of nearest-neighbour spin interactions and a uniform magnetic field (controlled by a parameter q) running through the lattice.

We should also consider the continuum limits of such models as the lattice spacing tends to zero. In many cases, one obtains a conformally invariant quantum field theory. Such 'conformal field theories' turned out to have their own rich algebraic structure (called 'vertex algebras'), and were connected with modular forms, the 'monster group' and other topics. One of them (the Wess–Zumino–Novikov–Witten model) underlies the quantum group knot invariants above.

Finally, we should consider the classical mechanical systems underlying the continuum limits of the exactly solvable lattice models. These turn out to be certain nonlinear, but completely integrable, partial differential equations. A typical feature of such equations was the presence of 'soliton' solutions, and a method of classical inverse scattering had earlier been developed to describe them (see Faddeev & Takhtajan 1987). Thus, one can trace a certain continuity of ideas through several key developments in mathematical physics.

(*c*) *Revolutions in Lie theory*

We have mentioned that the concept of a Lie algebra g is one of the most central and beautiful in mathematics and physics. The definition is innocent enough: a vector space g and a 'Lie bracket' map $[\cdot,\cdot]$ sending $g\otimes g\to g$ and obeying

$$[x,x]=0,\qquad [x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0, \tag{4.2}$$

for all $x,y,z\in g$, but the resulting theory of such objects is deep and extensive. Originally due to Sophus Lie, E. Cartan and others in the late 19th century, this theory of Lie algebras has itself undergone a rejuvenation in the last two decades because of quantum groups.

First of all, we have already given the example of sl_2 and its relation with quantum groups in § 3. By analysing this and, in general, how the quantum groups $U_q(g)$ behave as $q\to 1$, we can add certain concepts to Lie theory that correspond to the tiny difference between $U_q(g)$ and the enveloping algebra of g when q is infinitesimally close to 1. The resulting concept of a *Lie bialgebra* consists of a usual Lie algebra g equipped with a 'Lie co-bracket' $\delta: g\to g\otimes g$. The latter is like an infinitesimal version of the co-product Δ of a quantum group, and implies, in particular, that for every Lie bialgebra g, there is a dual g^*. These ideas (due to Drinfeld (1983)) have allowed mathematicians to go back and understand many hard constructions in conventional Lie theory in a more elegant and natural manner, as well as to obtain entirely new results.

Also, one can formally regard quantum groups such as $SL_{q,2}$ as some kind of 'quantization' (even though they are not really the algebra of observables of a true quantum system and q need not be related to Planck's constant). The classical mechanical system is necessarily described by a so-called Poisson bracket on the underlying classical group. These Poisson brackets have remarkable properties connected to the 'complete integrability' of the models. They have led to many developments in the differential geometry of classical Lie groups themselves.

Finally, there is an important subalgebra $U_q(n_-) \subseteq U_q(g)$, which can be used to generate representations of the whole quantum group from a 'vacuum' vector. Lusztig (1993) introduced a nice description of this subalgebra in terms of certain advanced concepts of 'perverse sheaves' and 'shifted' perverse sheaves from algebraic geometry. Without going into details, he obtained in this way a basis of the subalgebra $U_q(n_-)$ with many remarkable integrality and positivity properties, called the *Kashiwara–Lusztig canonical basis*. The most remarkable property of this basis is that it induces a basis of every representation that $U_q(n_-)$ generates. This might seem esoteric, but all these results continue to hold even when $q = 1$, and as such they provided unsuspected and revolutionary results in the representation theory of ordinary Lie algebras g themselves. There is also some non-commutative geometry behind the canonical basis, which remains largely unexplored.

5. Braided algebra, a calculus for non-commutative geometry

Coming out of the deeper structure of quantum groups is a particular brand of non-commutative geometry, called *braided geometry* (Majid 1991*a*, 1995). In fact, its inspiration is not so much the 'inner' non-commutativity within one algebra (as motivated by quantization) but the 'outer' non-commutativity between any two independent algebras. For example, when we consider a box of photons or other identical bosonic particles in quantum theory, we symmetrize the wave function under all permutations of the particles. For a box of electrons or identical fermionic particle, we 'skew-symmetrize' with a -1 factor whenever two particles are exchanged. It turns out that a deeper point of view on the parameter q in quantum groups (different from thinking of it as related to Planck's constant) is that q (or the matrix R) plays the role of this -1. We say that such systems have 'braid statistics'. We have already explained in § 4 that the representations of quantum groups are intrinsically braided, i.e. anything on which a quantum group acts acquires braid statistics. In effect, the usual division in nature into particles of force (bosons) and the building blocks of matter (fermions) is blurred in non-commutative geometry, and interrelated with quantization.

This idea is far reaching. Most constructions in mathematics involve applying a sequence of operations or maps. We can think of these as 'boxes' with some inputs and some outputs, and make more complex computations by 'wiring up' the outputs of one into the inputs of another, much as one wires up the silicon chips in a computer. The big difference in *braided mathematics* is that when this wiring up requires us to cross our wires, we allow a non-trivial operation R, typically different for an under- or an over-crossing. This is shown in figure 6*a*, where the product on an algebra B is depicted simply as a joining of two copies of B to result in one copy. Information flows down the page, i.e. we 'scan' the diagram from top to bottom. The figure expresses associativity and the concept of an identity element as a map η. Likewise, the axioms for the braided-mathematical version of a group, i.e. a 'braided group', are shown in

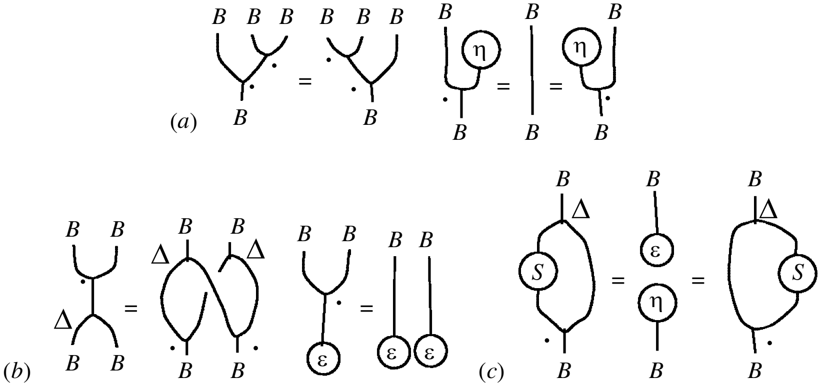

Figure 6. Axioms of a braided group as diagrams.

the rest of the figure. As for quantum groups, there is a 'co-product' that branches from one to two copies. It obeys the axioms in figure 6*a* turned upside down.

It turns out that most general constructions in group theory and quantum group theory go through in this braided setting. For example, the notion of conjugation is shown in figure 7. On the left-hand side is a 'breakdown' of the steps involved in usual conjugation in a group. One can think of a group trivially as a braided group with a co-product that simply doubles up the group element. We double up h in this way, move it past the g, apply the antipode or inversion operation S and then multiply up. The corresponding diagram is shown on the right. In this way, one arrives at a theory of algebras and groups that exists entirely at the level of braids and branches. One can do proofs, roughly speaking, by treating these as actual strings, i.e. this is a kind of knot-theoretic algebra.

For example, the algebra $B = \mathbb{C}[x]$ of polynomials in one variable x is a commutative algebra (so this has nothing to do with quantization) and can be regarded as a braided group with

$$\Delta x = x \otimes 1 + 1 \otimes x, \qquad \epsilon x = 0, \qquad Sx = -x, \tag{5.1}$$

which expresses the additive group law on the line. But, because we regard it as braided, there is a factor q, say, when one copy $\mathbb{C}[y]$ of the algebra moves past another independent copy $\mathbb{C}[x]$. This is the correct point of view in the formula (1.1) for the q-deformed derivative. Its origin is therefore not exactly quantum groups but the more primitive idea of braid statistics. Similarly, the quantum plane B generated by x, y as in (3.1) is not in any natural way a quantum group but it is precisely a braided group. It has an additive co-product as in (5.1) but with braiding defined by a matrix R. It is the same as the one used in constructing the knot invariant associated to $U_q(sl_2)$ in § 4. So the same circle of ideas allowed Jackson's q-analysis, in the 1990s, to be generalized from one to many variables. Such quantum-braided planes in turn feature in the deeper structure of quantum groups $U_q(g)$ (Majid 1999*a*).

Finally, having the proper tools, one can press on and develop further aspects of non-commutative geometry. First of all, one needs 'differential forms' for which the q-deformed derivatives are the associated partial derivatives. On a general algebra M, one can specify an abstract exterior algebra of differential forms $(\mathrm{d}, \Omega(M))$ obeying certain properties, notably that $\mathrm{d}^2 = 0$. The main difference is that we do not assume

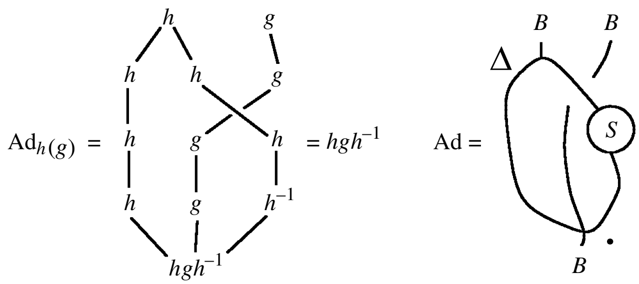

Figure 7. Conjugation written as a diagram.

that differentials such as dx commute with coordinates such as x. Building on this, the non-commutative versions of the main notions in differential geometry, such as ‘principal bundles’ and ‘gauge theory’, but with quantum group symmetry were introduced in the early 1990s (Brzeziński & Majid 1993). Without going into details, one arrives at a proposal for a ‘quantum (Riemannian) manifold’ (Majid 1999b) as an algebra M equipped with a differential calculus, a frame quantum principal bundle, and, for a Riemannian quantum manifold, a second dual frame quantum principal bundle. This is not yet quantum gravity, but it is, at least, one approach to a good part of the required mathematical infrastructure.

6. A new philosophical basis for the 21st century

We return now to our opening theme, the problem of the unification of quantum theory and gravity. This is summarized in figure 8, which plots the mass energy and size of many objects in the Universe. The key point here is that everything to the left is forbidden by quantum mechanics, which says that particles are also waves of a wavelength inversely proportional to the mass energy. Everything to the right is forbidden by Einstein’s theory of gravity, which says that as one puts more mass into a given volume it forms a black hole of a size proportional to the mass. Where these two overlap, namely at masses and distances around

$$m_{\mathrm{Planck}} = \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-5}\ \mathrm{g}, \qquad x_{\mathrm{Planck}} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-33}\ \mathrm{cm},$$

both theories break down. It is certainly possible to imagine phenomena near the bottom of the ‘vee’ that could test a complete theory of quantum gravity.

Taking another tack, as we look further and further out into space we see the Universe as it was earlier and earlier in time, since it has taken light longer to reach us. From this, a fairly conventional picture has emerged; that as we go back in time the Universe was smaller and smaller and gravitational fields stronger and stronger. Hence, at some point, both quantum effects and gravitational effects will be strong and our extrapolation will start to break down. Any predictions before that point, i.e. about where the Universe came from, really have to await a theory of quantum gravity. Conversely, astronomical data could provide a testing ground for such a theory. For example, just as light travelling over solar distances provided the first tests of Einstein’s theory, high-energy gamma rays travelling over cosmological distances could provide a test of quantum geometry, any further deviations predicted by that being amplified over the vast distances involved.

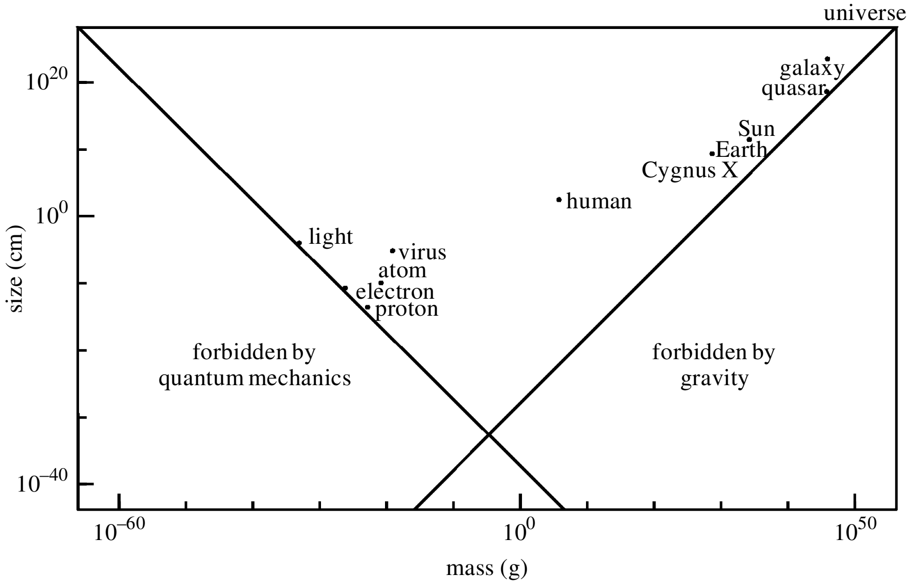

Figure 8. Size versus mass energy of objects in the Universe.

We conclude this article by stepping back and asking if there are at least some philosophical insights to be gained from all this. After several centuries of the pursuit of science or 'natural philosophy' and within sight of the 'holy grail' of quantum gravity, are we any nearer to answering the deeper questions of existence? We will now argue that we are, although it has to be said that what follows is strictly the author's personal view, expressed a decade ago in Majid (1991*b*).

To begin with, theoretical physics usually takes the reductionist view that there are indeed some fundamental laws of nature, of which our experiments and observations are representations. Thus it is supposed that something is absolutely true, and that something else measures or observes it. However, one of the themes of modern mathematics is that such evaluations should generally be thought of more symmetrically as a 'duality pairing' of one structure with another. An evaluation $f(x)$ can also be read $x(f)$, where f is an element of a dual structure. Since theoretical physics adopts the language of mathematics, such an 'observer–observed' reversed interpretation of the mathematical structure can always be forced, but will the dual interpretation also look like physics? We postulate that this should be so as a general *principle of representation-theoretic self-duality*, that a fundamental theory of physics is incomplete unless such a role reversal is possible. We can go further and hope to fully determine the (supposed) structure of fundamental laws of nature among all mathematical structures by this self-duality condition.

Such duality considerations are certainly evident in some form in the context of quantum theory and gravity. The structural reasons, from a mathematical point of view, are summarized to the left in figure 9. For example, Lie groups provide the simplest examples of Riemannian geometry, while the representations of similar Lie groups provide the quantum numbers of elementary particles in quantum theory. More generally, both quantum theory and the theory of curved spaces are needed for a self-dual picture, and, in general terms, Einstein's equation for gravity does indeed

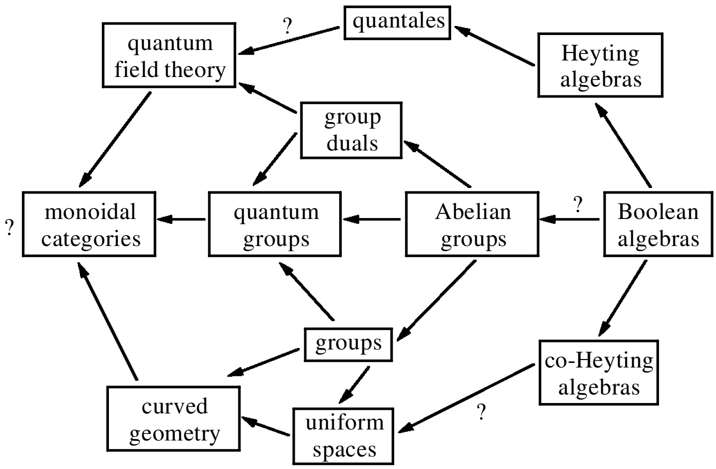

Figure 9. Representation-theoretic approach to quantum gravity.

equate a geometrical object (the Einstein tensor, which measures the curvature of space-time) to a quantum-mechanical object (the vacuum expectation of the stress energy tensor, which measures the matter content), i.e. it can be interpreted as some kind of self-duality constraint. The diagram also suggested (a decade ago) that the unifying language for both quantum theory and gravity would be category theory itself, a very general setting bordering on semantics.

Quantum groups provide a simple setting where the unification is indeed possible. They are a self-dual class of objects that unify both groups (curved spaces) and group duals (quantum numbers), as we have seen to some extent in earlier sections. One can then ask for self-dual quantum groups as a constraint on the structure of physics, leading immediately to the Planck-scale quantum group $\mathbb{C}[\mathbf{x}] \blacktriangleright\!\!\triangleleft \mathbb{C}[\mathbf{p}]$ in § 3 with features somewhat like a black hole. The representation-theoretic duality has a clear physical meaning in this context as an observer–observed dualism. The dual quantum group contains the 'states' ϕ, as explained in § 2, but, by the self-duality, someone could equally well think of ϕ as an observable and write $\phi(\mathbf{x})$ as $\mathbf{x}(\phi)$.

In addition to the 'end' of theoretical physics in the form of quantum gravity, we can ask also about its 'birth' in Aristotelean logic and the scientific method. The structure is that of Boolean algebras, and these too are self-dual under the interchange of AND with OR (an operation implemented via the NOT operation). They are on the right in figure 9. Going above the axis takes us to Heyting algebras and on into intuitionistic logic, where one drops the law that either a proposition or its negation is true. This is also the essential feature of the logical structure of quantum mechanics. Dual to this is the notion of co-Heyting algebra and co-intuitionistic logic, in which one drops the axiom that the intersection of a proposition and its negation is empty. It has been argued by F. W. Lawvere and his school that this intersection is like the 'boundary' of the proposition, and, hence, that these co-Heyting algebras are the 'birth' of geometry. Moreover, both geometry and quantum theory take us off the self-dual axis. This is also confirmed by physics: in Aristotelean logic (the simplest theory of physics), we can regard a chair or a 'not-chair' as equally good concepts, but this self-duality is lost, for example, in a theory of gravity alone (a chair curves space, a not-chair does not). If the above is correct, the person who regarded not-chairs as real should instead interpret gravity as a quantum effect and vice versa (with self-duality being restored only in a combined theory of quantum gravity). Thus, when we say space is almost a vacuum except for quantum effects (the left slope in figure 8), the other person would say that space is almost full of not-chairs (from their perspective the right slope in figure 8).

In between these extremes, a well-known example of a self-dual setting is that of Abelian groups such as the real line. For every (reasonable) such group, there is a dual one $G\hat{}$ of its representations. Especially important is that $G\hat{}\hat{}$ is isomorphic to G. The position and momentum groups in flat space are dual to each other in this way and Fourier transform precisely allows us to transform our point of view from one to the other. This familiar setting gives a clue to the philosophical basis of the principle of representation-theoretic self-duality. For, if some theorist thought that a group G was the 'true' structure underlying a law of physics and that $G\hat{}$ was its representations, a more experimentally minded physicist might equally well consider $G\hat{}$ as the true object and G as its representations. Only in a self-dual setting could both points of view be entertained. This says that the principle has its origins in the nature of the scientific method. And, if Einstein's equation and other laws of theoretical physics could be deduced from such a principle alone, we would have achieved a Kantian or Hegelian view of the nature of physical reality as a consequence of the choice to look at the world in a certain way through logic and the scientific method. Of course, it is never going to be that simple, but it is something to think about on a rainy day in the next millennium.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Born in Patna, India, but moving to the UK at the age of 5, Shahn Majid completed his BA in 1982 with a first class degree in mathematics at Cambridge, followed in 1983 by a distinction in Part III. He pursued a PhD at Harvard with the help of an Emmanuel–Harvard Herchel Smith scholarship, completing it in 1988. After a year at Swansea, he returned to Cambridge in 1989 as Drapers Fellow of Pembroke College, where he has been based since, aside from a two-year leave back at Harvard in 1995 and 1996. In 1993 he was awarded the Bleuler Medal. He has many research papers and one textbook in the area of quantum groups. Aged 38, he is an associate editor of two research journals and a Royal Society University Research Fellow.

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